Chaos and Time-Series Analysis

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Preface

To be written

1

2

2.1

Change the variable X_n to $X_n = \left(\frac{Y_n}{A} + \frac{1}{2}\right)$. Substitution in (2.4) yields: $\frac{Y_{n+1}}{A} + \frac{1}{2} = A\left(\frac{Y_n}{A} + \frac{1}{2}\right)\left(1 - \frac{Y_n}{A} - \frac{1}{2}\right) = A\left(\frac{1}{4} - \frac{Y_n^2}{A^2}\right)$. Multiply both sides by A to get : $Y_{n+1} = B - Y_n^2$ with $B = \frac{A^2}{4} - \frac{A}{2}$

2.2

Fixed point: $X^* = 1 - 1/2.8 \simeq 0.643$
The first ten iterations are:
$x_1 = 0.631$, $x_2 = 0.652$, $x_3 = 0.635$, $x_4 = 0.649$, $x_5 = 0.638$, $x_6 = 0.647$
$x_7 = 0.640$, $x_8 = 0.645$, $x_9 = 0.641$, $x_{10} = 0.644$ which indicate an oscillation about X^*

2.3

Let $f(X) = AX(1-X)$ and consider $X_0 = X^* + \varepsilon_0$, with ε_0 very small and positive. Then $X_1 = f(X_0) = AX^* - AX^{*2} - 2AX^*\varepsilon_0 + A\varepsilon_0 - A\varepsilon_0^2$.
X^*satisfies $X^* = AX^* - AX^{*2}$ which gives $X_1 = X^* - 2AX^*\varepsilon_0 + A\varepsilon_0 - A\varepsilon_0^2$.
The $-A\varepsilon_0^2$ term is smaller than the ε_0 terms and may be neglected. Hence $X_1 = X^* + \varepsilon_0 A(1-2X^*) = X^* + \varepsilon_0 f'(X^*)$. X_1 may be written as $X_1 = X^* + \varepsilon_1$
thus giving $\varepsilon_1 = \varepsilon_0 f'(X^*)$. After n iterations: $\varepsilon_n = \varepsilon_0 \left(f'(X^*)\right)^n$
Hence the fixed point is unstable when $|f'(X^*)| > 1$. For $f'(X^*) > 1$ all ε's are positive which means $X \to +\infty$ as $n \to +\infty$. This is not the case for negative slopes since the ε's can both be positive and negative depending whether n is even or odd.

2.4

See text on page 17.

2.5

The second iterate map is given by $g(X) = f(f(X))$ with $f(X) = AX(1-X)$.Let p and q be the fixed points of the map (these two points are given by eqn 2.7).

$g'(X)|_{X=p} = f'(f(X))f'(X)|_{X=p} = f'(f(p))f'(p) = f'(q)f'(p)$
$= A^2(1-2p)(1-2q) = A^2[1-2(p+q)+4pq]$
From eqn 2.6 observe that $p+q = [A(A+1)]/A^2$ and $pq = (A+1)/A^2$
Hence $g'(X)|_{X=p} = A^2[1-2A(A+1)/A^2 + 4\,(A+1)\,/A^2] = 4+2A-A^2$
Eqn 2.8 is found by setting $g'(X) = -1$.In general the 2-cycle is stable when $|g'(X)| < 1$ ie when $3 < A < 1+\sqrt{6}$

2.6

?

2.7

The map $f(X) = 4X(1-X)$ has pre-images $X_{A,B} = \dfrac{1}{2}(1 \pm \sqrt{1-X})$. Then $f'(X_{A,B}) = \mp 4\sqrt{1-X}$ and $P(X_{A,B}) = \dfrac{2}{\pi\sqrt{X}}$. Use these two equations in eqn 2.12 to obtain $P(X)$ as in eqn 2.11

2.8

?

2.9

The pre-images of $X_{n+1} = f(X_n) = A\min(X_n, 1-X_n)$ are $X_A = X/A$ and $X_B = 1 - X/A$. Use these together with the fact that $|f'(X_{A,B})| = 1$ in eqn 2.12 to obtain $P(X) = 1/A$.

2.10

Let $f(X) = A\,(1-|2X-1|^{\alpha})$. Then $f'(X) = -2\alpha A\mathrm{sign}(2X-1)\,|2X-1|^{\alpha-1}$. Obviously $f'\,(1/2)$ is discontinuous for $\alpha \leq 1$ (not defined for $\alpha < 1$ and having a jump discontinuity for $\alpha = 1$). The second derivative is given by: $f''(X) = -2\alpha\,(\alpha-1)\,A(\mathrm{sign}(2X-1))^2\,|2X-1|^{\alpha-2} = -2\alpha\,(\alpha-1)\,A\,|2X-1|^{\alpha-2}$. $f''(1/2)$ is not defined for $\alpha < 2$, but for $\alpha \geq 2$, f'' is continuous everywhere and therefore f is smooth (Note: $\mathrm{sign}(x) = 1$ for $x \geq 0$ and -1 for $x < 0$).

2.11

From Exercise 2.10 the derivative at 0 is given by $f'(0) = 2\alpha A\,|-1|^{\alpha-1}$. Since $\alpha < 1/2$ and $0 < A < 1$ we may conclude that $|f'(0)| < 1$ and hence that the point X=0 is stable. The existence of a stable point eliminates the possibility of presence of chaos in the map.

3

3.1

The solution to eqn 3.1 is of the form $x_1 = x_0 e^{at}$, where x_0 is the initial condition. A nearby trajectory starting from $x_0 + \varepsilon$ with ε small is given by $x_2 = (x_0 + \varepsilon)\, e^{at}$. Their separation $\Delta x = x_2 - x_1 = \varepsilon e^{at}$ increases exponentially at a rate a.

3.2

The solution $x = \dfrac{1}{1 + \left(\dfrac{1}{x_0} - 1\right) e^{-at}}$ has derivative $\dfrac{dx}{dt} = \dfrac{a\left(\dfrac{1}{x_0} - 1\right) e^{-at}}{\left[1 + \left(\dfrac{1}{x_0} - 1\right) e^{-at}\right]^2}$.

Use x and its derivative in the expression $\dfrac{dx}{dt} - ax\,(1 - x)$. Verifying that this expression equals 0 proves that eqn 3.3 satisfies eqn 3.2.

3.3

This equation is a separable 1st order ODE. It can be written in the form $\csc(x)\, dx = dt$, but the solution can be given only in implicit form. Integrating both sides and applying the initial condition $x(0) = x_0$ we obtain :
$\ln\left[(\csc x - \cot x) / (\csc x_0 - \cot x_0)\right] = t \Leftrightarrow \csc x - \cot x = (\csc x_0 - \cot x_0)\, e^t$

3.4

$dx/dt = ay$ and $dy/dt = -bx$. Differentiate the first equation with respect to time and substitute dy/dt to get $d^2x/dt^2 + abx = 0$. This can have a solution of the form $x = x_0 \sin\left(\sqrt{ab}t\right)$. Substitution in the second equation yields $dy/dt = -bx_0 \sin\left(\sqrt{ab}t\right)$. Integrating once we get $y = x_0\sqrt{\frac{b}{a}}\cos\left(\sqrt{ab}t\right)$. The equations for x and y generate points that lie on an ellipse with $\frac{x_{\max}}{y_{\max}} = \sqrt{\frac{a}{b}}$.

3.5

$v\left(t\right)=\dfrac{dx}{dt}=v_0\cos\omega t+\dfrac{A\Omega}{\omega^2-\Omega^2}\cos\Omega t$ and $\dfrac{d^2x}{dt^2}=-\omega v_0\sin\omega t-\dfrac{A\Omega^2}{\omega^2-\Omega^2}\sin\Omega t.$
Using the expressions for x and d^2x/dt^2in the left hand side of ean 3.10 we finally obtain $A\sin\Omega t$ which equals the right hand side.

3.6

$\frac{d^2x}{dt^2}=-v_0\frac{-\omega^4e^{-\omega^2\frac{t}{b}}+b^4e^{-bt}}{b^3}$ and $\frac{dx}{dt}=v_0\frac{-\omega^2e^{-\omega^2\frac{t}{b}}+b^2e^{-bt}}{b^2}$. Let $\Delta=\frac{d^2x}{dt^2}+b\frac{dx}{dt}+\omega^2x=-v_0\omega^2\frac{-\omega^2e^{-\omega^2\frac{t}{b}}+b^2e^{-bt}}{b^3}$.Then comparing Δ with each term from eqn 3.18 we get $\left|\Delta/\frac{d^2x}{dt^2}\right|=\frac{\omega^2e^{-\omega^2\frac{t}{b}}-b^2e^{-bt}}{b^2e^{-\omega^2\frac{t}{b}}-b^2e^{-bt}}\ll\frac{\omega^2e^{-\omega^2\frac{t}{b}}-b^2e^{-bt}}{\omega^2e^{-\omega^2\frac{t}{b}}-b^2e^{-bt}}=1$, $\left|\Delta/\left(b\frac{dx}{dt}\right)\right|=\frac{\omega^2}{b^2}\ll\frac{1}{4}$ and $\left|\Delta/\left(\omega^2x\right)\right|=\frac{-\omega^4e^{-\omega^2\frac{t}{b}}+b^2\omega^2e^{-bt}}{-\omega^4e^{-\omega^2\frac{t}{b}}+b^4e^{-bt}}\ll\frac{-\omega^4e^{-\omega^2\frac{t}{b}}+b^4e^{-bt}}{-\omega^4e^{-\omega^2\frac{t}{b}}+b^4e^{-bt}}=1$. Since each term is much less than 1 we may neglect Δ, thus verifying that the eqn 3.21 satisfies eqn 3.18 .

3.7

$x=v_0te^{-bt/2}$, $v(t)=dx/dt=v_0e^{-\frac{1}{2}bt}-\frac{1}{2}v_0tbe^{-\frac{1}{2}bt}$ and $d^2x/dt^2=-v_0be^{-\frac{1}{2}bt}+\frac{1}{4}v_0tb^2e^{-\frac{1}{2}bt}$. Substitute these in the right hand side of eqn 3.18 to verify that it equals zero.

3.8

$x=\left(v_0/\omega\right)e^{-\frac{1}{2}bt}\sin\omega t$, $v(t)=dx/dt=-\frac{1}{2}v_0e^{-\frac{1}{2}bt}\frac{b\sin\omega t-2(\cos\omega t)\omega}{\omega}$ and $d^2x/dt^2=\frac{1}{4}v_0e^{-\frac{1}{2}bt}\frac{b^2\sin\omega t-4b(\cos\omega t)\omega-4(\sin\omega t)\omega^2}{\omega}$. Substitution in eqn 3.18 yields
$\Delta=d^2x/dt^2+bdx/dt+\omega^2x=-\frac{1}{4}v_0e^{-\frac{1}{2}bt}b^2\frac{\sin\omega t}{\omega}$. Compare Δ with each term from eqn 3.18 to get : $\left|\Delta/\left(\omega^2x\right)\right|=\dfrac{\frac{1}{4}v_0e^{-\frac{1}{2}bt}b^2\frac{\sin\omega t}{\omega}}{v_0\omega e^{-\frac{1}{2}bt}\sin\omega t}=\frac{1}{4}\frac{b^2}{\omega^2}\ll1$,
$\left|\dfrac{\Delta}{b\frac{dx}{dt}}\right|=\dfrac{\frac{1}{4}v_0e^{-\frac{1}{2}bt}b^2\left|\frac{\sin\omega t}{\omega}\right|}{\frac{1}{2}v_0be^{-\frac{1}{2}bt}\left|\frac{b\sin\omega t-2(\cos\omega t)\omega}{\omega}\right|}<\dfrac{\frac{1}{4}b^2\left|\frac{\sin\omega t}{\omega}\right|}{b}=\frac{1}{4}b\left|\frac{\sin\omega t}{\omega}\right|\leq\frac{1}{4}\dfrac{b}{\omega}\ll1$ and $\left|\Delta/\frac{d^2x}{dt^2}\right|=\dfrac{\frac{1}{4}v_0e^{-\frac{1}{2}bt}b^2\left|\frac{\sin\omega t}{\omega}\right|}{\frac{1}{4}v_0e^{-\frac{1}{2}bt}\left|\frac{b^2\sin\omega t-4b(\cos\omega t)\omega-4(\sin\omega t)\omega^2}{\omega}\right|}<\dfrac{b^2\left|\frac{\sin\omega t}{\omega}\right|}{\left|4\omega\sin\omega t\right|}=\frac{1}{4}\frac{b^2}{\omega^2}\ll1$. Since Δ is much smaller than each of the terms in eqn 3.18 we may consider the eqn 3.23 roughly satisfying eqn 3.18.

3.9

$x=\frac{A\sin(\Omega t-\phi)}{\sqrt{(\omega^2-\Omega^2)^2+b^2\Omega^2}}$, $v(t)=dx/dt=\frac{A\Omega\cos(\Omega t-\phi)}{\sqrt{(\omega^2-\Omega^2)^2+b^2\Omega^2}}$and $d^2x/dt^2=-\frac{A\Omega^2\sin(\Omega t-\phi)}{\sqrt{(\omega^2-\Omega^2)^2+b^2\Omega^2}}$.
Substitution in the right hand side of eqn 3.17 yields:

$$\frac{d^2x}{dt^2}+b\frac{dx}{dt}+\omega^2x=A\frac{\left(\omega^2-\Omega^2\right)\sin(\Omega t-\phi)+b\Omega\cos(\Omega t-\phi)}{\sqrt{(\omega^2-\Omega^2)^2+b^2\Omega^2}}\qquad(*).$$

Use eqn 3.25 and the trigonometric identities $\cos\phi=1/\sqrt{1+\tan^2\phi}=$

$\frac{\omega^2-\Omega^2}{\sqrt{(\omega^2-\Omega^2)^2+b^2\Omega^2}}$ and $\sin\phi = \sqrt{1-\cos^2\phi} = \frac{b\Omega}{\sqrt{(\omega^2-\Omega^2)^2+b^2\Omega^2}}$. Eqn (*) can then be written as:
$d^2x/dt^2+bdx/dt+\omega^2x = A\left(\cos\phi\sin\left(\Omega t-\phi\right)+\sin\phi\cos\left(\Omega t-\phi\right)\right) = A\sin\Omega t.$

3.10

$x_m = \frac{A}{\sqrt{(\omega^2-\Omega^2)^2+b^2\Omega^2}}$ and the derivative $\frac{dx_m}{d\omega} = \frac{-A\Omega\left(-2\omega^2+2\Omega^2+b^2\right)}{\left(\sqrt{(\omega^4-2\omega^2\Omega^2+\Omega^4+b^2\Omega^2)}\right)^3}$. It becomes zero for $\Omega = \frac{1}{2}\sqrt{(4\omega^2-2b^2)}$ and $\Omega = -\frac{1}{2}\sqrt{(4\omega^2-2b^2)}$. x_m becomes maximum for $\Omega = \frac{1}{2}\sqrt{(4\omega^2-2b^2)}$ and the maximum is $\frac{2A}{b\sqrt{(4\omega^2-b^2)}}$.

3.11

Find the Ω's for which $\frac{x_m}{A} = \frac{1}{\sqrt{(\omega^2-\Omega^2)^2+b^2\Omega^2}} = \frac{1}{\sqrt{2}b\omega} \Leftrightarrow$
$2b^2\omega^2 = \left(\omega^2-\Omega^2\right)^2 + b^2\Omega^2$. Then $\Omega^4 + \left(b^2-2\omega^2\right)\Omega^2 + \omega^4 - 2b^2\omega^2 = 0 \Leftrightarrow (\Omega/\omega)^4 + \left((b/\omega)^2-2\right)(\Omega/\omega)^2 + 1 - 2(b/\omega)^2 = 0$. The Ω's satisfy $\left(\frac{\Omega_1}{\omega}\right)^2 + \left(\frac{\Omega_2}{\omega}\right)^2 = 2 - \frac{1}{Q^2}$ and $\left(\frac{\Omega_1}{\omega}\right)^2\left(\frac{\Omega_2}{\omega}\right)^2 = 1 - \frac{2}{Q^2} \Rightarrow \frac{\Omega_1}{\omega}\frac{\Omega_2}{\omega} \simeq 1 - \frac{1}{Q^2}$ for large Q. Hence $\left(\frac{\Delta\Omega}{\omega}\right)^2 = \left(\frac{\Omega_1}{\omega}\right)^2 + \left(\frac{\Omega_2}{\omega}\right)^2 - 2\frac{\Omega_1}{\omega}\frac{\Omega_2}{\omega} \simeq 2 - \frac{1}{Q^2} - 2\left(1-\frac{1}{Q^2}\right) = \frac{1}{Q^2} \Rightarrow \frac{\Delta\Omega}{\omega} = \frac{1}{Q}$

3.12

Start with $x^2+v^2 = x^2+\left(\frac{dx}{dt}\right)^2 \simeq 4$ and take the derivative at each side to obtain $2x\frac{dx}{dt}+2\frac{dx}{dt}\frac{d^2x}{dt^2} \simeq 0 \Leftrightarrow x+\frac{d^2x}{dt^2} \simeq 0$ which corresponds to eqn 3.26 in the limit as $b \to 0$.

3.13

Use $v=\frac{dx}{dt}$ to write eqn3.26 as $x+\frac{dv}{dt}+bv\left(v^2+x^2-1\right) = 0 \Rightarrow$
$vx+v\frac{dv}{dt}+bv^2\left(v^2+x^2-1\right) = 0 \Rightarrow \frac{1}{2}\frac{d}{dt}\left(x^2+v^2\right)+bv^2\left(v^2+x^2-1\right) = 0$
which is apparently satisfied by $v^2+x^2 = 1$, the limit cycle.

3.14

?

3.15

SAME AS Prob 3.13

3.16

$f = x-y-x^3$ and $g = x-x^2y$. At $(0,0)$ we have $dx/dt = dy/dt = 0$ and the Jacobian evaluated there is:

$$J(0,0)=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$$

The eigenvalues can then be found to be $\lambda = 0$ and 1 from which we may conclude that the origin is at an unstable equilibrium. For solutions approximately on the unit circle ie satisfying $x^2+y^2\simeq 1$ we have $dx/dt \simeq x\left(1-x^2\right)-y = xy^2-y$. Multiply dx/dt by x and dy/dt by y to get:
$x\left(dx/dt\right)\simeq x^2y^2-xy$ and $y\left(dy/dt\right)=yx-x^2y^2$. Adding those together we get $x\left(dx/dt\right)+y\left(dy/dt\right)=\frac{1}{2}\frac{d}{dt}\left(x^2+y^2\right)\simeq 0$. Therefore the points near the unit circle tend to remain close to it since the rate of change of their distance from the origin is close to zero.

3.17

Write eqn 3.18 as a system of 1st order ODEs ie $\frac{dy}{dt}=-by-\omega^2x=-y-x=g(x,y)$ and $\frac{dx}{dt}=y=f(x,y)$. The equations corresponding to 3.33 and 3.34 with $h=1$ are $x_{n+1}=x_n+y_n$ and $y_{n+1}=-x_n$. These two can be written in a single equation $x_{n+1}=x_n-x_{n-1}\quad(*)$. Starting with $x_0=y_1=\beta$ and $x_1=\alpha$ iterate $(*)$ to get $x_2=\alpha-\beta$, $x_3=-\beta$, $x_4=-\alpha$, $x_5=-\alpha+\beta$, $x_6=\beta=x_0$.

3.18

Total error: $e(h,t,\epsilon)=th+\epsilon\sqrt{\frac{t}{h}}\Rightarrow\frac{\partial e}{\partial h}=t-\frac{1}{2}\epsilon\sqrt{\frac{t}{h^3}}$. For minimum set $\frac{\partial e}{\partial h}=0$ to get $h=\sqrt[3]{\frac{\epsilon^2}{4t}}$. For this value of h , $\frac{\partial^2 e}{\partial h^2}>0$ which guarantees that $h=\sqrt[3]{\frac{\epsilon^2}{4t}}$ gives the minimum.

3.19

The equations become $x_{n+1}=x_n+h\left(\frac{y_{n+1}+y_n}{2}\right)$ and $y_{n+1}=y_n+h\left(-\frac{x_{n+1}+x_n}{2}\right)$ which give $x_1=x_0+h\left(\frac{y_1+y_0}{2}\right)$ and $y_1=y_0-h\left(\frac{x_1+x_0}{2}\right)$. Solving this system we obtain $y_1=\frac{-4h}{4+h^2}$ and $x_1=\frac{4-h^2}{4+h^2}$. These give $y_1^2+x_1^2=\frac{\left(4-h^2\right)^2+16h^2}{4+h^2}=1$, so the error is zero.

3.20

$\frac{dy}{dt}=-by-\omega^2x=g(x,y)$ and $\frac{dx}{dt}=y=f(x,y)$. Then :
$x_{n+1}=x_n+hf\left(\frac{x_{n+1}+x_n}{2},\frac{y_{n+1}+y_n}{2}\right)$ and $y_{n+1}=y_n+hg\left(\frac{x_{n+1}+x_n}{2},\frac{y_{n+1}+y_n}{2}\right)$ which give $x_{n+1}=x_n+h\frac{y_{n+1}+y_n}{2}$ and $y_{n+1}=y_n+h\left(-b\frac{y_{n+1}+y_n}{2}-\omega^2\frac{x_{n+1}+x_n}{2}\right)$.

3.21

If we use the leapfrog method to the x variable once the same results are obtained as in the text on page 46, namely $x_1=1$ and $y_1=-h$ which as before gives an error of the order of h^2 since $\Delta R=h^2/2$. However when we

use the leapfrog method to the y variable we get $x_1 = 1 - h^2$ and $y_1 = -h$ which give
$x_1^2 + y_1^2 = 1 - h^2 + h^4 \simeq 1 - h^2$. We also get an error of the order of h^2, but in this case $\Delta R = -h^2/2$.

3.22

Let $P_n = (X_n, Y_n)$ and $P_0 = (1, 0)$. This problem is solved in two ways:
1. Using the leapfrog method in the x variable:
The equations become: $X_{n+1} = X_n + Y_n$ and $Y_{n+1} = Y_n - X_{n+1}$. Then $P_1 = (1, -1)$, $P_2 = (0, -1)$, $P_3 = (-1, 0)$, $P_4 = (-1, 1)$, $P_5 = (0, 1)$, $P_6 = P_0 = (1, 0)$.
2. Using the leapfrog method in the y variable:
The equations are: $Y_{n+1} = Y_n - X_n$ and $X_{n+1} = X_n + Y_{n+1}$ and we get $P_1 = (-1, 0)$, $P_2 = (-1, -1)$, $P_3 = (0, -1)$, $P_4 = (1, 0)$, $P_5 = (1, 1)$, $P_6 = P_0 = (1, 0)$.
In either case we still get a solution of period 6 which is comparable to the period $2\pi \simeq 6.28$ of the exact solution.

3.23

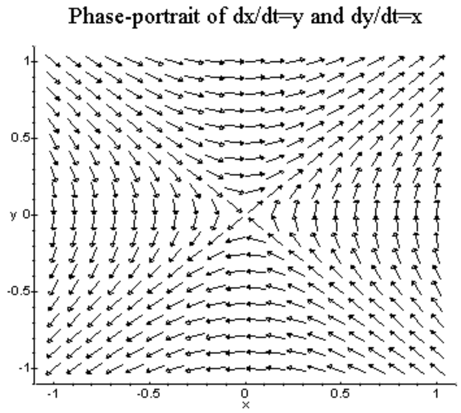
?

3.24

?

4

4.1



4.2

Expand eqn 4.5 to get $\lambda^2 - (a + d)\,\lambda + ad - bc = 0$ and then use the quadratic formula to obtain the two eigenvalues.

4.3

DONE IN TEXTBOOK ON PAGE 58

4.4

The Jacobian is given by: $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ whose characteristic polynomial is $\lambda^2 + 1 = 0$. Hence $\lambda = \pm i$

4.5

The eigenvalues given by eqn 4.12 can be written as $\lambda_{1,2} = -\frac{b}{2}\left(1 \pm \sqrt{1 - \left(\frac{2\omega}{b}\right)^2}\right)$.

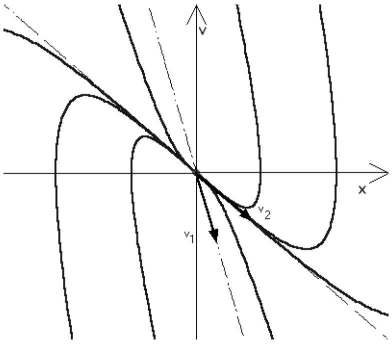
The term under the square root can never exceed 1, so the two eigenvalues can either be complex conjugates or real with the same sign. Thus a saddle point cannot occur since this requires λ_1 and λ_2 to be real with opposite signs. For $b > 2\omega$ we get stable nodes $(\lambda_1 < \lambda_2 < 0)$ if b is positive and unstable nodes if b is negative $(\lambda_1 > \lambda_2 > 0)$. When $b = 2\omega$ we either get a degenerate stable node (for $b > 0$) or a degenerate unstable node (for $b < 0$). When $b < 2\omega$ the term in the square root becomes negative thus giving complex solutions. Under this condition we get an unstable focus for $b < 0$ and a stable focus for $b > 0$.

4.6

If the arrows are in the direction towards the spiral and radial points, each of these points are stable, otherwise they are unstable. Doing the same on the saddle point has no effect on its overall stability.

4.7

In order to find the eigenvectors v one must solve the equation $(J - \lambda I)\, v = 0$ ie $\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & b-\lambda \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ The solution is $v_i = \begin{pmatrix} 1 \\ \lambda_i \end{pmatrix}$, $i = 1, 2.$ For $b = 2.5\omega > 0$, $\lambda_1 = -4b/5$ and $\lambda_2 = -b/5$ we obtain the figure shown below which is similar to the middle plot of Fig 4.1.



4.8

The Jacobian $J = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ gives the characteristic equation $(1-\lambda)(4-\lambda) + 2 = 0$ with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$ and eigenvectors $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4.9

Add the two solutions given by 4.6 to obtain $\lambda_1 + \lambda_2 = a + d =$trace$(J)$ and multiply them to get $\lambda_1\lambda_2 = ad - bc =det(J)$.

4.10

The Jacobian of the damped harmonic oscillator has trace$(J) = -b$ and $\det(J) = \omega^2$. For the critically damped case $b = 2\omega$. Hence trace$(J) = \pm 2\sqrt{\det(J)}$.

4.11

Denote by τ the trace of J and by δ its determinant. The eigenvalues given by eqn 4.6 may be written as $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$. The ultimate fate for each romantic style can be determined by the values of τ and δ. Therefore we can have the following cases:
CASE 1: $\tau > 0$ and $\tau^2 > 4\delta > 0$. In this case both λ_1 and λ_2 are positive which means that the origin is an unstable node and therefore the couple may end up extremely loving or hating each other or each of them having opposite feelings for each other. What decides their final state is a matter of the initial conditions.
CASE 2: $\tau > 0$ and $\tau^2 < 4\delta$. In such a case λ_1 and λ_2 are complex conjugates with positive real part which means that the origin is an unstable spiral. No matter what the initial conditions are the couple will never be indifferent for each other. They will go through cycles of growing love and growing hate forever (ie oscillatory feelings of growing intensity) , never reaching some final feeling for his/her mate.
CASE 3. $\tau = 0$ and $\delta > 0$. The origin becomes a center. In this case the couple will pass through never- ending periods of love and hate, but these feelings are oscillatory with fixed maximum intensity and do not lead to such an "explosion" of feelings as in the cases 1 and 2. Moreover the couple will never be indifferent for each other like in cases 1 and 2.
CASE 4: $\delta < 0$ The origin is a saddle.Depending on the values of the parameters a, b, c and d we can have two cases:
4a. The couple will either end up extremely loving each other or extremely hating each other.
4b. The couple will have growing opposite feelings for each other (eg Romeo extremely loving her and Juliet extremely hating him or vice versa).
Since the eigenvectors may be written as $v_i = \begin{pmatrix} \frac{b}{a-\lambda_i} \\ -1 \end{pmatrix}$ with $i = 1, 2$, we end up in case 4a if for the positive eigenvalue we have $\frac{b}{a-\lambda} < 0$ and in case 4b if this expression is positive for the positive eigenvalue.
The outcomes of 4a and 4b are dependent upon the initial feelings for each other. In both cases 4a and 4b there is a possibility of that Romeo and Juliet become completely indifferent for each other. This may happen if and only if initially the couple has feelings that lie on the eigendirection whose eigenvalue is negative.
CASE 5: $\tau < 0$. In this case the couple will end up being indifferent for one another. The way this happens depends on whether we have $\tau^2 > 4\delta > 0$, where we get a stable node or $\tau^2 < 4\delta$, where we get a stable spiral.

4.12

IMPOSSIBLE TO MAKE SUCH PLOTS. The trace and determinant are not sufficient to give the nature of the eigenvalues. For the 2×2 case this was possible since knowing the sum and product of the eigenvalues suffice to fully determine the eigenvalues.

4.13

The equation $\lambda^3 + 2.14\lambda^2 + 1 = 0$ is solved using a computer and we obtain $\lambda_1 = -2.33$ and $\lambda_{2,3} = 0.0925 \pm 0.649i$ which verifies that the origin is a spiral saddle of index 2.

4.14

The equilibria can be found by solving $dx/dt = dy/dt = dz/dt = 0$ using eqns 4.33-4.35. It can be shown that the solutions are $p_1^* = (0,0,0)$ and $p_{2,3}^* = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. Obviously the first bifurcation occurs when $r = 1$ at which the equilibria p_2^* and p_3^* are created. The Jacobian is
$J(x,y,z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$ which gives the characteristic equations $(b+\lambda)(\lambda^2 + (\sigma+1)\lambda + \sigma(r-1) = 0$ for p_1^* and $\lambda^3 + (\sigma+b+1)\lambda^2 + b(\sigma+r)\lambda + 2b\sigma(r-1) = 0$ for $p_{2,3}^*$. For $1 < r < r^*$ the equilibria are spiral nodes and for $r > r^*$ they become unstable spiral saddle points with index 2 (see text on page 69). Since this happens, at the point where $r = r^*$ the two of the three eigenvalues should be complex conjugates with no real part. Assuming solutions of the form $\lambda = i\omega$ substitute in the characteristic equation for $p_{2,3}^*$. The real and imaginary parts of the left hand side should both equal to zero which give $\omega^2 = b\,(\sigma + r^*)$ and $\omega^2\,(\sigma+b+1) = 2b\sigma\,(r^*-1)$ respectively. Substituting ω^2 in the second equation and solving for r^* we obtain $r^* = \dfrac{\sigma(b+\sigma+3)}{\sigma-b-1}$

4.15

Solving the system of equations $dx/dt = dy/dt = dz/dt = 0$ we obtain two equilibria, namely $p_{1,2}^* = (\mp 1, \pm 1, 0)$. The Jacobian is given by
$J(x,y,z) = \begin{pmatrix} -1 & -1 & 0 \\ -z & 0 & -x \\ y & x & 0 \end{pmatrix}$. Hence $J(p_{1,2}^*) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & \pm 1 \\ \pm 1 & \mp 1 & 0 \end{pmatrix}$. For both p_1^* and p_2^* the Jacobian has the same characteristic polynomial $\lambda^3 + \lambda + \lambda^2 + 2 = 0$. Solving this equation using a computer we get $\lambda_1 = -1.35$ and $\lambda_{2,3} = 0.177 \pm 1.20i$.

4.16

Use eqns 4.36-4.38 to solve $dx/dt = dy/dt = dz/dt = 0$. These give $x^* = az^*$, $y^* = -z^*$ and $z^* = \frac{c \pm \sqrt{c^2 - 4ab}}{2a}$. The Jacobian is given by $J(x,y,z) =$

$\begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{pmatrix}$, whose characteristic equation is:
$\lambda^3 - \lambda^2 \left(\frac{b}{z^*} + a\right) + \lambda \left(z^* + \frac{ba}{z^*} + 1\right) - az - \frac{b}{z^*} = 0.$

5

5.1

Working as in section 5.1.1 on page 81 the result is $\lambda =$ln 10

5.2

Since we have $|f'(x_n)| = A > 0$ we get $\lambda = \ln|A|$ which is negative for $0 < A < 1$ so there exists a stable fixed point for $0 < A < 1$. For $A > 1$ we always have $|f'(x^*)| > 0$ at $x = x^*$ so it is impossible to have stable orbits.

5.3

NOT SURE ABOUT REASONING. After many iterations the map settles to either a fixed point or a cycle. At bifurcation points, all points which are fixed or belong to a cycle have their derivatives equal to 1 and hence the logarithm of their derivatives equal to zero. Since the Lyapunov exponent equals to the average of those logarithms after many iterations (see eqn 5.3), we may conclude that the Lypunov exponent equals to zero.

5.4

Since there is only one fixed point $X^* = 1 - 1/A$ the probability distribution may take the form $P(X) = \delta(X - X^*)$, where δ is the Dirac delta function. So using $\lambda = \int_0^1 P(X) \ln|A(1-2X)|\, dX$ we obtain $\lambda = \ln|A(1-2X^*)| = \ln|2-A|$.

5.5

Start from $g(X) = f(f(X)) = A^2X - A^2\left(A^2+1\right)X^2 + 2A^3X^3 - A^3X^4$ which gives $g'(X) = A^2 - 2A^2\left(A^2+1\right)X + 6A^3X^2 - 4A^3X^3$. Then the probability distribution is given by $P(X) = \frac{1}{2}\left[\delta(X - X_+^*) + \delta(X - X_-^*)\right]$ with $X_\pm^*$ being given in eqn 2.7. Use $\lambda = \int_0^1 P(X)\ln|g'(X^*)|\, dX$ to obtain $\lambda = \frac{1}{2}\ln\left|g'(X_+^*)g'(X_-^*)\right|$. A lot of algebra would be required to evaluate this expression, but there is an easier way for doing so if we consider $g'(X_\pm^*) = f'(f(X_\pm^*))f'(X_\pm^*) = f'(X_\mp^*)f'(X_\pm^*) = A^2(1-2X_\mp^*)(1-2X_\pm^*) = A^2[1 - 2(X_\mp^* + X_\pm^*) + 4X_\mp^*X_\pm^*]$. From equation (2.6) $X_+^* + X_-^* = [A(A+1)]/A^2$ and $X_-^*X_+^* = (A+1)/A^2$. Hence $g'(X_\pm^*) = A^2[1 - 2A(A+1)/A^2 + 4\,(A+1)\,/A^2] = 4 + 2A - A^2$ and therefore $\lambda = \ln\left|4 + 2A - A^2\right|$.

5.6

The fixed point for $A = 2$ is $X^* = 1-1/2 = 1/2$. Moreover since $f(X^*) = 0$ we may conclude that there exists a supercycle since these results would force the Lyapunov exponent to tend to $-\infty$. Iterating the map starting from $X_0 = 0.1$ we get $X_1 = 0.2952$, $X_2 = 0.41611392$, $X_3 = 0.4859262512$, $X_4 = 0.4996038592$, $X_5 = 0.4999996862$ and $X_6 = 0.5000000000$.

5.7

Substitution of $A = 1+\sqrt{5}$ in eqn 2.7 we get $X_+^* = \frac{3+\sqrt{5}}{2(1+\sqrt{5})}$ and $X_-^* = \frac{1}{2}$. In order to verify that we have supercycle we have to show that $g'\left(X_-^*\right) = 0$ with g' being given in the solution of Problem 5.5. With a bit of algebra one can verify that $g'\left(X_\pm^*\right) = 0$.

5.8

???????

5.9

Since the map for those range of values of α has a single fixed point the Lyapunov exponent equals to $\lambda = \ln|f'(X^*)|$. Since $X^* = 0$ (the fixed point) we may write $\lambda = \ln|f'(0)| = \ln(2\alpha)$.

5.10

$$\frac{\Delta R_{n+1}^2}{\Delta R_n^2} = \frac{(a\Delta X_n + b\Delta Y_n)^2 + (c\Delta X_n + d\Delta Y_n)^2}{\Delta X_n^2 + \Delta Y_n^2} = \frac{\left(a + bY_n^{'}\right)^2 + \left(c + dY_n^{'}\right)^2}{1 + Y_n^{\prime 2}}$$

with $Y_n^{'} = \dfrac{\Delta Y_n}{\Delta X_n} = \dfrac{c\Delta X_{n-1} + d\Delta Y_{n-1}}{a\Delta X_{n-1} + b\Delta Y_{n-1}} = \dfrac{c + dY_{n-1}^{'}}{a + bY_{n-1}^{'}}$ which is the same as eqn 5.12. Substitution of $\frac{\Delta R_{n+1}^2}{\Delta R_n^2}$ in eqn 5.10 yields eqn 5.11.

5.11

Using the notation of the text we get $|a| = A$, $b = c = 0$ and $|d| = B$. Substitution in eqn 5.11 yields $\lambda_1 = \lim\limits_{N\to\infty} \dfrac{1}{2N} \sum\limits_{n=0}^{N-1} \ln\left[\frac{A^2+B^2Y_n^{\prime 2}}{1+Y_n^{\prime 2}}\right] = \dfrac{1}{2} \lim\limits_{n\to\infty} \ln\left[B^2 + \frac{A^2-B^2}{1+Y_n^{\prime 2}}\right]$ with $\left|Y_n^{'}\right| = \left|\frac{BY_{n-1}^{'}}{A}\right| = |Y_0| \left|\frac{B}{A}\right|^n$. Then we have : $\lim\limits_{n\to\infty} \left|Y_n^{'}\right| = \begin{cases} +\infty & \mathit{for} \quad A < B \\ 0 & \mathit{for} \quad A > B \end{cases}$ Substitution of the limit in the equation for λ_1 gives $\lambda_1 = \ln[\max(A, B)]$. Also use $\lambda_1 + \lambda_2 = \det(J) = |AB|$ to determine λ_2.

5.12

A.1.1 Logistic Map : $J(X) = A(1-2X)$
A.1.2 Sine Map : $J(X) = A\pi\cos\pi X$
A.1.3 Tent Map : $J(X) = A\ \mathrm{sign}(1/2-X)$
A.1.4 Linear Congruential Generator : $J(X) = A(\mathsf{mod}\ C)$
A.1.5 Gauss Map : $J(X) = -1/X^2(\mathsf{mod}\ 1)$
A.2.1 Hènon Map : $J(X,Y) = \begin{pmatrix} -2aX & b \\ 1 & 0 \end{pmatrix}$
A.2.2 Lozi Map : $J(X,Y) = \begin{pmatrix} -a\ \mathrm{sign}(X) & b \\ 1 & 0 \end{pmatrix}$
A.2.3 Tinkerbell Map : $J(X,Y) = \begin{pmatrix} 2X+a & -2Y+b \\ 2Y+c & 2X+d \end{pmatrix}$
A.2.4 Burgers Map : $J(X,Y) = \begin{pmatrix} a & -2Y \\ Y & X+b \end{pmatrix}$
A.2.5 Kaplan-Yorke Map : $J(X,Y) = \begin{pmatrix} a & 0 \\ -4\pi\sin(4\pi X) & b \end{pmatrix}(\mathsf{mod}\ 1)$
A.2.6 Dissipative Standard Map: $J(X,Y) = \begin{pmatrix} 1 & 1 \\ k\cos X(\mathsf{mod}\ 2\pi) & b \end{pmatrix}$
A.3.1 Chirikov Map : $J(X,Y) = \begin{pmatrix} 1 & 1 \\ k\cos(X) & 1 \end{pmatrix}(\mathsf{mod}\ 2\pi)$
A.3.2 Hènon Area-preserving Quadratic Map : $J(X,Y) = \begin{pmatrix} \cos\alpha + 2X\sin\alpha & -\sin\alpha \\ \sin\alpha - 2X\cos\alpha & \cos\alpha \end{pmatrix}$
A.3.3 Arnold Cat Map: $J(X,Y) = \begin{pmatrix} 1 & 1 \\ 1 & k \end{pmatrix}(\mathsf{mod}\ 1)$
A.3.4 Gingerbreadman Map : $J(X,Y) = \begin{pmatrix} \mathsf{sign}(X) & -1 \\ 1 & 0 \end{pmatrix}$
A.3.5 Stochastic Web Map : $J(X,Y) = \begin{pmatrix} \cos\alpha - k\sin\alpha\cos X & -\sin\alpha \\ \sin\alpha + k\cos\alpha\cos X & \cos\alpha \end{pmatrix}$
A.3.6 Lorenz Simplest 3-D Map : $J(x,y,z) = \begin{pmatrix} Y & X & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

5.13

Using eqn 5.20 we get $\lambda = a\,(1-2x)$ for the local Lyapunov exponent. Since at $x^* = 1$ we have $f\,(x^*) = 0$ the global Lyapunov exponent is $\lambda = -a$.

5.14

5.15

Since $J(x,y) = \begin{pmatrix} 0 & 1 \\ -1-2bxy & b(1-x^2) \end{pmatrix}$, we have $\det(J) = 2bxy + 1$. For area expansion we require $\det(J) > 0$, ie $xy > -1/2b$ and for area contraction we must have $xy < -1/2b$.

5.16

INCOMPLETE

We have $a = d = 0$, $b = c = 1$. From eqn 5.23 we get $\lambda_1 + \lambda_2 = 0$ and from eqn 5.21 $\lambda_1 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{2y'}{1+y'^2} dt$ with $\frac{dy'}{dt} = 1 - y'^2$ for $\Delta t \to 0$

$\int \frac{1}{1-y^2} dy = arctanh\ y$

$\int \frac{2\tanh t}{1+\tanh^2 t} dt = -\frac{1}{2}\ln(-1+\tanh t) - \frac{1}{2}\ln(1+\tanh t) + \frac{1}{2}\ln\left(1+\tanh^2 t\right)$

$\lim_{t\to\infty} \left(-\frac{1}{2}\ln(-1+\tanh t) - \frac{1}{2}\ln(1+\tanh t) + \frac{1}{2}\ln\left(1+\tanh^2 t\right)\right) = \infty$

5.17

5.18

Since $\frac{\partial}{\partial x}[\sigma(y-x)] + \frac{\partial}{\partial y}[rx - y - xz] + \frac{\partial}{\partial z}[xy - bz] = -b - 1 - \sigma < 0$ for all $\sigma, b > 0$ we have volume contraction with $V(t) \sim e^{-(b+1+\sigma)t}$.

5.19

A.4.1 Damped Driven Pendulum :

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ -\cos x & -b & A\cos\Omega z \\ 0 & 0 & 0 \end{pmatrix}$$

A.4.2 Driven Van Der Pol Oscillator :

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ -1-2bxy & b(1-x^2) & A\cos\Omega z \\ 0 & 0 & 0 \end{pmatrix}$$

A.4.3 Ueda Attractor :

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ -3x^2 & -b & A\cos\Omega z \\ 0 & 0 & 0 \end{pmatrix}$$

A.5.1 Lorenz Attractor :

$$J(x,y,z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

A.5.2 Rössler Attractor :

$$J(x,y,z) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{pmatrix}$$

A.5.3 Diffusionless Lorenz Attractor :

$$J(x,y,z) = \begin{pmatrix} -1 & -1 & 0 \\ -z & 0 & -x \\ y & x & 0 \end{pmatrix}$$

A.5.4 Chua's circuit :

$$J(x,y,z) = \begin{pmatrix} \alpha(b-1+1/2(a-b)(\mathsf{sign}(x+1)-\mathsf{sign}(x-1)) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}$$

A.5.5 Moore-Spiegel Oscillator :

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -T-2Rxy & -(T-R+Rx^2) & -1 \end{pmatrix}$$

A.5.6 Thomas' Cyclically Symmetric Attractor :

$$J(x,y,z) = \begin{pmatrix} -b & \cos y & 0 \\ 0 & -b & \cos z \\ \cos x & 0 & -\beta \end{pmatrix}$$

A.5.7 Simplest Quadratic Chaotic Flow :

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2y & -a \end{pmatrix}$$

A.5.8 Simplest Cubic Chaotic Flow :

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ y^2-1 & 2yx & -a \end{pmatrix}$$

A.5.9 Simplest Piecewise Linear Chaotic Flow :

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathrm{sign}(x) & -1 & -a \end{pmatrix}$$

A.6.1 Simplest Driven Chaotic Flow :

$$J(x,y,z)=\begin{pmatrix} 0 & 1 & 0 \\ 3x^2 & 0 & \cos\Omega z \\ 0 & 0 & 0 \end{pmatrix}$$

A.6.2 Nosé-Hoover Oscillator :

$$J(x,y,z)=\begin{pmatrix} 0 & 1 & 0 \\ -1 & z & y \\ 0 & -2y & 0 \end{pmatrix}$$

A.6.3 Labyrinth Chaos :

$$J(x,y,z)=\begin{pmatrix} 0 & \cos y & 0 \\ 0 & 0 & \cos z \\ \cos x & 0 & 0 \end{pmatrix}$$

A.6.4 Hènon-Heilles System :

$$J(x,y,v,w)=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1-2y & -2x & 0 & 0 \\ -2x & 2y-1 & 0 & 0 \end{pmatrix}$$

5.20

A. $\sum\lambda=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^T zdt=\langle z\rangle$ Numerical calculation is needed.
B,C,E,I,L,M,R,S. $\sum\lambda=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^T -1dt=-1$
D,O. $\sum\lambda=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^T xdt=\langle x\rangle$ Numerical calculation is needed.
F,H,Q. $\sum\lambda=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^T (0.5-1)\,dt=-0.5$
G. $\sum\lambda=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^T (0.4-1)\,dt=-0.6$
J,N. $\sum\lambda=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^T -2dt=-2$
K. $\sum\lambda=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^T (y-1+0.3)\,dt=\langle y\rangle-0.7$ Numerical calculation is needed.
P. $\sum\lambda=\lim\limits_{T\to\infty}\dfrac{1}{T}\int_0^T (2y)\,dt=2\langle y\rangle$Numerical calculation is needed.

5.21

Since one of the eigenvalues is positive we have a strange attractor of dimension greater than 2.

5.22

$D_{KY}=3+\dfrac{1}{27}(0.11+0.02)=3.004\,8$

5.23

$D_{KY} = 4 + \frac{1}{0.6}(0.5 + 0.1 + 0 - 0.3) = 4.5$

5.24

We first have to find the equation of a parabola $y = ax^2 + bx + c$ which passes through the points $(0, p_1)$, $(1, p_2)$ and $(2, p_3)$. Solving the resulting system of three equations in three unknowns we get

$$y = \left(\frac{1}{2}p_1 - p_2 + \frac{1}{2}p_3\right)x^2 + \left(4p_2 - \frac{3}{2}p_3 - \frac{5}{2}p_1\right)x + p_3 - 3p_2 + 3p_1.$$

Hence we get :
A.5.1 $p_1 = p_2 = 0.9056, p_3 = -13.6667$ and dimension $\simeq 2.111$
A.5.2 $p_1 = p_2 = 0.0714$, $p_3 = -5.3229$ and dimension $\simeq 2.025$
A.5.3 $p_1 = p_2 = 0.2101$, $p_3 = -1$ and dimension $\simeq 2.273$
A.5.4 $p_1 = p_2 = 0.3271$, $p_3 = -2.1926$ and dimension $\simeq 2.213$
A.5.5 $p_1 = p_2 = 0.1109$, $p_3 = -1$ and dimension $\simeq 2.171$
A.5.6 $p_1 = p_2 = 0.0349$, $p_3 = -0.54$ and dimension $\simeq 2.109$
A.5.7 $p_1 = p_2 = 0.0551$, $p_3 = -2.017$ and dimension $\simeq 2.051$
A.5.8 $p_1 = p_2 = 0.0837$, $p_3 = -2.028$ and dimension $\simeq 2.074$
A.5.9 $p_1 = p_2 = 0.0362$, $p_3 = -0.6$ and dimension $\simeq 2.103$
A.6.2 $p_1 = p_2 = 0.0138$, $p_3 = 0$ and dimension $= 3$
A.6.3 $p_1 = p_2 = 0.1402$, $p_3 = 0$ and dimension $= 3$.

5.25

5.26

$\lambda = \ln(a(\mathsf{mod} c))$

6

6.1

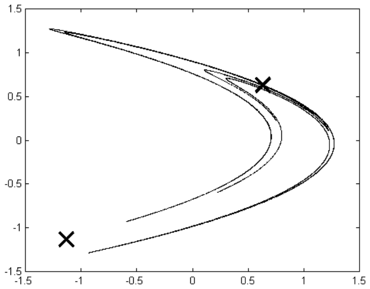
In order to determine the inverse map interchange X_n and X_{n+1}, Y_n and Y_{n+1} to get $Y_n = bX_{n+1}$ and $X_n = 1+Y_{n+1} - aX_{n+1}^2$. Solving for Y_{n+1} and X_{n+1} gives the inverse map. Answer: $X_{n+1} = Y_n/b$, $Y_{n+1} = X_n + aY_n^2/b - 1$.

6.2

The fixed points satisfy $X_{n+1} = X_n$ and $Y_{n+1} = Y_n$. By eqn 6.2, this means that $X_{n+1} = X_n = Y_{n+1} = Y_n = X^* = Y^*$. Eqn 6.1 becomes $(a_4 + a_5 + a_6)\, X^{*2} + (a_2 + a_3 - 1)\, X^* + a_1 = 0$. This equation has real solutions when
$\Delta = (a_2 + a_3 - 1)^2 - 4a_1\,(a_4 + a_5 + a_6) \geq 0$. Under this condition the fixed points are given by $X^* = Y^* = \left(1 - a_2 - a_3 \pm \sqrt{\Delta}\right) / \left(2a_4 + 2a_5 + 2a_6\right)$.

6.3

The fixed points of the Hénon Map are given by the equation $1.4X^2 + 0.7X - 1 = 0$, whose solution is $X = -1.131\,4$ and $X = 0.631\,35$. Their position relative to the attractor is shown in the figure below:



6.4

Interchange X_n and X_{n+1}, Y_n and Y_{n+1} to get : $X_n = a_1 + a_2X_{n+1} + a_3Y_{n+1} + a_4X_{n+1}^2 + a_5X_{n+1}Y_{n+1}$ and $X_{n+1} = Y_n$. Solving for Y_{n+1} and X_{n+1} gives the inverse map. Answer: $X_{n+1} = Y_n$, $Y_{n+1} = \frac{X_n - a_1 - a_2Y_n - a_4Y_n^2}{a_3 - a_5Y_n}$.

6.5

As in the previous problem use $X_{n+1} = Y_n$ and $X_n = a_1 + a_2 X_{n+1} + a_3 Y_{n+1} + a_4 X_{n+1}^2 + a_5 X_{n+1} Y_{n+1} + a_6 Y_{n+1}^2$. The second equation becomes $a_6 Y_{n+1}^2 + (a_5 Y_n + a_3) Y_{n+1} + X_n - a_1 - a_2 Y_n - a_4 Y_n^2 = 0$, whose solution is:

$$Y_{n+1} = \frac{1}{2a_6}\left[-a_5 Y_n - a_3 \pm \sqrt{(a_5 Y_n + a_3)^2 - 4a_6 \left(X_n - a_1 - a_2 Y_n - a_4 Y_n^2\right)}\right]$$

6.6

Antisymmetry about $r = 0.5$ can be verified if we notice that $f_i(0.5 - r) = -f_i(0.5 + r)$ for $i = 1, 2, 3$. It is obvious that f_1 is infinite for $r = 0$ and $r = 1$ since $\tan(\pm\pi/2) = \pm\infty$. Morever , one can verify that eqns 6.4 and 6.5 are also infinite using the Del' Hôpital's rule. All three functions have half their values in the range $-1 < f < 1$ since we have $f(0.5 \pm 0.25) = \pm 1$.

6.7

6.8

Eqn 6.6 may be also written as a system of differential equations of the form: $\dot{x} = y$, $\dot{y} = z$ and $\dot{z} = -\frac{3}{5}z - y + |x| - 1$. Equilibrium points satisfy $\dot{x} = \dot{y} = 0$. This happens only when $y = z = 0$ and $x = \pm 1$. The Jacobian takes the form:

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathrm{sign}(x) & -1 & -a \end{pmatrix}$$

When $x = -1$ the eigenvalues are $-0.58846, 0.59423 \pm 1.1603i$ and when $x = 1$ the eigenvalues are $0.83555, -0.11778 \pm 1.0876i$.

6.9

The following calculations are valid as long as the a_4 and a_5 are positive numbers. Eqn 6.17 becomes $\frac{\alpha}{\beta^2}\frac{dx'^2}{dt'^2} + a_1\left(\alpha^2 x^2 - a_4\right)\frac{\alpha}{\beta}\frac{dx'}{dt'} + a_5 \alpha x' = a_2 \sin(a_3 t') \Rightarrow$
$\frac{dx'^2}{dt'^2} + a_1 \beta \alpha^2 \left(x^2 - \frac{a_4}{\alpha^2}\right)\frac{dx'}{dt'} + a_5 \beta^2 x' = \frac{\beta^2}{\alpha} a_2 \sin(a_3 \beta t')$. Choose β so that $a_5 \beta^2 = 1$ ie $\beta = \pm\sqrt{a_5^{-1}}$ and $\alpha = \pm\sqrt{a_4}$. In such a case the equation becomes $\frac{dx'^2}{dt'^2} + a_1'\left(x^2 - 1\right)\frac{dx'}{dt'} + x' = a_2' \sin(a_3' t')$ with $a_1' = \pm a_1 a_4 \sqrt{a_5^{-1}}$, $a_2' = \pm\frac{a_2}{a_5\sqrt{a_4}}$ and $a_3' = \pm a_3\sqrt{a_5^{-1}}$.

6.10

In this problem we have to find the a 's and b's for which $X_n = b_1 \tanh(a_{10} + a_{11} X_{n-1}) + b_2 \tanh(a_{20} + a_{21} X_{n-1})$. The information we have in order to solve the problem is that when X_{n-1} is 0 or 1 then $X_n = 0$ and that the maximum should be at $X_n = 1$ when $X_{n-1} = 1/2$. Since we have six unknowns and 4 equations we may have two free variables. The equations are: $b_1 \tanh(a_{10}) + b_2 \tanh(a_{20}) = b_1 \tanh(a_{10} + a_{11}) + b_2 \tanh(a_{20} + a_{21}) = 0$, $b_1 a_{11} - b_1 a_{11} \tanh^2(a_{10} + a_{11}/2) + b_2 a_{21} - b_2 a_{21} \tanh^2(a_{20} + a_{21}/2) = 0$ and $b_1 \tanh(a_{10} + a_{11}/2) + b_2 \tanh(a_{20} + a_{21}/2) = 1$

6.11

Cannot determine the LE accurately enough from the diagram in order to compare them with those in the appendix.

6.12

Since we are interested only for the maximum, fit the parabola $x = at^2 + bt + c$ through the points $(-1, x_1)$, $(0, x_2)$ and $(1, x_3)$. In this case we would get $a = \frac{1}{2}(x_1 + x_3)$, $b = \frac{1}{2}(x_3 - x_1)$ and $c = x_2$. The maximum of the parabola occurs at $t = -b/(2a)$ and equals to $x_{\max} = c - b^2/4a$.

6.13

The intersection with the Poincaré section occurs only when $z_1 z_2 < 0$. Each point (x, y, z) on the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $x = x_1 + \lambda(x_2 - x_1)$, $y = y_1 + \lambda(y_2 - y_1)$ and $z = z_1 + \lambda(z_2 - z_1)$ with $0 \le \lambda \le 1$. For $\lambda = \dfrac{z_1}{z_1 - z_2}$ we have $z = 0$, ie an intersection of the line with the plane $z = 0$. Therefore the coordinates of the point (x, y) on the Poincaré section are given by $x = x_1 + \dfrac{z_1(x_2 - x_1)}{z_1 - z_2}$ and $y = y_1 + \dfrac{z_1(y_2 - y_1)}{z_1 - z_2}$.

6.14

6.15

6.16

6.17

6.18

The fixed points of the Lozi map satisfy $X^* = Y^*$ and $0.5X^* = 1 - 1.7|X^*|$. This gives either $0.5X^* = 1 - 1.7X^* \Rightarrow X^* = Y^* = 0.45455$ or $0.5X^* = 1 + 1.7X^* \Rightarrow X^* = Y^* = -0.83333$.

6.19

The fixed points satisfy $X^* = 10X^{*2} - 10Y^{*2} - 6Y^*$ and $X^* = \dfrac{Y^*}{4(Y^* + 1)}$. Solving these equations we finally obtain the pairs $(0, 0)$ and $(1.0276, -1.3215)$.

6.20

The fixed points satisfy $X = X^2 - Y^2 - 1$ and $Y(1-2X) = 0 \Rightarrow (i)$ $Y = 0$ and (ii) $X = 1/2$. Then for $Y = 0$ (case (i))we get the equation $X^2 - X - 1 = 0$ whose solutions are $X_1^* = \frac{1}{2}\sqrt{5} + \frac{1}{2}$ and $X_2^* = \frac{1}{2} - \frac{1}{2}\sqrt{5}$. The Jacobian of the map has characteristic polynomial $(2X-\lambda)^2 + 4Y^2 = 0$ and eigenvalues $\lambda = 2X \pm 2Yi$. Therefore X_1^* is unstable since $\lambda > 0$, whereas X_2^* is stable. For $X = 1/2$ we get non-real Y (case (ii)) . In order to determine the periodic cycle consider the fixed points of the second iterate map. For the Y component we get $4XY\left(X^2 - Y^2 - 1\right) = Y \Rightarrow$ $Y\left[4X\left(X^2 - Y^2 - 1\right) - 1\right] = 0$. The equation $4X\left(X^2 - Y^2 - 1\right) - 1 = 0$ together with the X component of the second iterate map do not have real solutions. However when $Y = 0$, using $\left(X^2 - Y^2 - 1\right)^2 - 4X^2Y^2 - 1 = X$ we get $X(X^3 - 2X - 1) = 0$. This gives $X = 0$ and $X = -1$. ,ie in this case we have a 2-period cycle with oscillations between the points $(0,0)$ and $(-1,0)$.

6.21

By definition, the derivative of f with respect to $\bar{Z}$ is $\dfrac{\partial f}{\partial \bar{Z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial X} + i\dfrac{\partial}{\partial Y}\right) f =$ $\dfrac{1}{2}\left(\dfrac{\partial F}{\partial X} - \dfrac{\partial G}{\partial Y}\right) + \dfrac{i}{2}\left(\dfrac{\partial G}{\partial X} + \dfrac{\partial f}{\partial Y}\right)$. Since f is a function of Z only we have $\dfrac{\partial f}{\partial \bar{Z}} = 0$. This condition requires $\dfrac{\partial F}{\partial X} = \dfrac{\partial G}{\partial Y}$ and $\dfrac{\partial G}{\partial X} = -\dfrac{\partial F}{\partial Y}$.

6.22

Hénon Map: $\dfrac{\partial F}{\partial X} = -2aX \neq 0 = \dfrac{\partial G}{\partial Y}$. Lozi map: $\dfrac{\partial F}{\partial Y} = 0.5 \neq 1 = \dfrac{\partial G}{\partial X}$. Julia set: $\dfrac{\partial F}{\partial X} = \dfrac{\partial G}{\partial Y} = 2X$ and $\dfrac{\partial G}{\partial X} = -\dfrac{\partial F}{\partial Y} = 2Y$.

6.23

The Cauchy-Riemann equations require $\dfrac{\partial F}{\partial X} = a_2 + 2a_4X + a_5Y = 0 = \dfrac{\partial G}{\partial Y}$. Hence we must have $a_2 = a_4 = a_5 = 0$. Also $\dfrac{\partial F}{\partial Y} = a_3 + a_5X + 2a_6Y =$ $-1 = -\dfrac{\partial G}{\partial X}$ which gives $a_3 = -1, a_5 = a_6 = 0$. Therefore in this case we must have $X_{n+1} = a_1 - Y_n$ and $Y_{n+1} = X_n$ which is obviously a non-chaotic system.

6.24

Let $F = b_1 + b_2X + b_3Y + b_4X^2 + b_5XY + b_6Y^2$ and $G = c_1 + c_2X +$ $c_3Y + c_4X^2 + c_5XY + c_6Y^2$. The Cauchy-Riemann equations require $\dfrac{\partial F}{\partial X} =$ $b_2 + 2b_4X + b_5Y \equiv c_3 + c_5X + 2c_6Y = \dfrac{\partial G}{\partial Y}$ and $\dfrac{\partial F}{\partial Y} = b_3 + b_5X + 2b_6Y \equiv$

$-c_2 - 2c_4X - c_5Y = -\dfrac{\partial G}{\partial X}$. Equating the coefficients with the same powers of X and Y we get : $b_2 = c_3 = a_2$, $b_4 = c_5/2 = a_4$, $b_5/2 = c_6 = a_5$, $b_3 = -c_2 = a_3$, $b_5/2 = -c_4 = a_5$ and $b_6 = -c_5/2 = -a_4$. If we also let $b_1 = a_1$ and $c_1 = a_6$ we obtain eqns 6.18 and 6.19.

7

7.1

This is done in the textbook on page 132. Eigenvalues?

7.2

Done in textbook (p.133). Eigenvalues?

7.3

When $y = 1$, $\dot{y} = 0$ for all values of the parameter a. Now, expand about $y = 1$ by making a change of variable from y to $v = y - 1$. In this case v is small and we have $\dot{y} = \dot{v} = a\ln(v+1) + v \approx a[v - \frac{1}{2}v^2 + ...] + v = (a+1)v - \frac{1}{2}av^2 + O(v^3)$. Hence a transcritical bifurcation occurs when $a = -1$. In order to write the last equation in normal form, let $v = c\xi$ which transforms it to $\dot{\xi} = (a+1)\xi - \frac{1}{2}ca\xi^2 + O(v^3)$. Letting $c = 2/a$ and $\mu = (a+1)$ we finally obtain the normal form with
$\xi = \frac{1}{2}a\,(y-1)\,.$

7.4

The fixed points satisfy $f = \mu x^* - x^{*3} = 0$. Solving for x^* we obtain $x^* = 0$ and $x^* = \pm\sqrt{\mu}$. Then $\frac{df}{dx^*} = \mu - 3x^{*2}$ and for $x^* = 0$ we get $\frac{df}{dx^*} = \mu$. Thus for $\mu > 0$, x^* is unstable and for $\mu < 0$ it is stable. However for the points $x^* = \pm\sqrt{\mu}$, which exist only for $\mu > 0$ we get $\frac{df}{dx^*} = -2\mu > 0$ which are stable.

7.5

The fixed points satisfy $f = \mu x^* + x^{*3} = 0$. Solving for x^* we obtain $x^* = 0$ and $x^* = \pm\sqrt{-\mu}$. Then $\frac{df}{dx^*} = \mu + 3x^{*2}$ and for $x^* = 0$ we get $\frac{df}{dx^*} = \mu$. Thus for $\mu > 0$, x^* is unstable and for $\mu < 0$ it is stable. However for the points $x^* = \pm\sqrt{-\mu}$, which exist only for $\mu < 0$ we get $\frac{df}{dx^*} = -2\mu > 0$ which are unstable.

7.6

For $a > 1$ the only equilibrium point we have is $x^* = 0$ which is unstable since $\frac{df}{dx^*} = a - \cos x^* > 0$. For $0 < a < 1$, the equilibrium points satisfy $\dfrac{\sin x^*}{x^*} = a$. INCOMPLETE-NEEDS FIGURE.

7.7

As shown in the text eqns 7.8 and 7.9 may be written in polar coordinates as $\frac{dr}{dt} = r\left(\mu - r^2\right) = f(r)$ and $\frac{d\theta}{dt} = 1$. The equation $f(r) = 0$ is satisfied for the points $r^* = \pm\sqrt{\mu}$ which exist for $\mu > 0$ and for $r^* = 0$. For $r^* = 0$ we have $\frac{df}{dr} = \mu$ which means that it is stable for $\mu < 0$ and unstable for $\mu > 0$. The points satisfying $r^* = \pm\sqrt{\mu}$ give $\frac{df}{dr} = -\mu$ which is always stable (given that $\mu > 0$).

7.8

Multiply eqn 7.8 by x and eqn 7.9 by y and add them together. Then $x\frac{dx}{dt} + y\frac{dy}{dt} = \frac{1}{2}\frac{dr^2}{dt} = r\frac{dr}{dt} = r^2\left(\mu - r^2\right)$.

7.9

The equation $\frac{d^2x}{dt^2} + b\left(x^2 - 1\right)\frac{dx}{dt} + x = a$ can be written as $\dot{x} = y$ and $\dot{y} = a - b\left(x^2 - 1\right)y - x$. The Jacobian of this system is thus:

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -2bxy - 1 & -b(x^2-1) \end{pmatrix}$$

The fixed points satisfy $\dot{x} = y = 0$ and $\dot{y} = 0$ which gives $x = a$. In such a case the characteristic polynomial is $\lambda\left[\lambda + b\left(a^2 - 1\right)\right] + 1 = 0$ which has solutions $\lambda = \frac{-b\left(a^2-1\right) \pm i\sqrt{4 - b^2\left(a^2-1\right)^2}}{2} = A(a,b) \pm iB(a,b)$. We have Hopf bifurcations when $b = 0$ since $A(a,0) = 0$, $B(a,0) \neq 0$ and $\frac{dA}{db} = -\frac{\left(a^2-1\right)}{2}$ which is greater than zero provided that $a < 1$. We may also have Hopf bifurcations when $a = \pm 1$ since $A(\pm 1, 0) = 0$, $B(\pm 1, b) \neq 0$ and $\frac{dA}{da} = \mp b$ which is greater than zero when b is positive for $a = -1$ and when b is negative for $a = 1$.

7.10

The equation $\frac{dr}{dt} = r(\mu + r^2 - r^4) = f(r)$ gives the fixed points $r_1^* = 0$ and $r_\pm^{*2} = \frac{1 \pm \sqrt{1+4\mu}}{2} r_{2,3}^{*2} = \frac{1 \pm \sqrt{1+4\mu}}{2}$. To classify their stability compute $\frac{df}{dr} = \mu + 3r^2 - 5r^4 = -4\mu - 2r^2 + 5(\mu + r^2 - r^4)$. Obviously for r_1^* $\frac{df}{dr} = \mu$ which is unstable for $\mu > 0$ and stable for $\mu < 0$. We have three cases : (i) $\mu < -1/4$ in which the only fixed point is r_1^* which is stable. (ii) $-1/4 < \mu < 0$ in which we find r_+^* to be stable and r_-^* to be unstable.(iii)$\mu > 0$. The cycle r_-^* vanishes, r_+^* is still stable whereas r_1^* becomes unstable.HYSTERESIS

7.11

HARD

7.12

Hilborn pages 257 and 258. Too hard to be included. At least a number of hints should be provided.

7.13

Write the second iterate map keeping the terms up to 3rd order : $F(F(X)) = -(1+\mu)\left[-(1+\mu)X + X^3\right] + \left[-(1+\mu)X + X^3\right]^3 \simeq X + 2X\mu + X\mu^2 - 2X^3 - 4X^3\mu - 3X^3\mu^2 - X^3\mu^3 = (1+\mu)^2 X - X^3 2 + 4\mu + 3\mu^2 + \mu \Rightarrow F(F(X)) \simeq (1+\mu)^2 X - (1+\mu)\left(2 + 2\mu + \mu^2\right) X^3$, which is identical to eqn 7.19. Then taking $F(F(\pm\sqrt{\mu})) = \pm(1+\mu)^2\sqrt{\mu} \mp \mu(1+\mu)\left(2+2\mu+\mu^2\right)\sqrt{\mu} = \pm\left[\sqrt{\mu} - 3\left(\sqrt{\mu}\right)^5 - 3\left(\sqrt{\mu}\right)^7 - \left(\sqrt{\mu}\right)^9\right]$. Since μ is small then we may neglect the 5th, 7th and 9th powers of $\sqrt{\mu}$ to get $F(F(\pm\sqrt{\mu})) \simeq \pm\sqrt{\mu}$ which shows that $\pm\sqrt{\mu}$ is approximately a fixed point.

7.14

Substitute $X^* = \ln A$ in $F(X) = AXe^{-X}$ to get $F(X^*) = X^*$, thus showing that X^* is a fixed point. In order to show the existence of the flip bifurcation make the change of variable $Y = X - \ln A$ and define $G(Y) = F(X) - X^*$to get $G(Y) = A(Y + \ln A)e^{-(Y+\ln A)} - \ln A = (Y + \ln A)e^{-Y} - \ln A$. The map $Y_{n+1} = G(Y_n)$ has a fixed point at $Y = 0$. Expand $G(Y)$ about 0 up to order 3 to get :
$G(Y) \simeq (Y + \ln A)\left(1 - Y + \frac{1}{2}Y^2 - \frac{1}{6}Y^3\right) - \ln A = (1 - \ln A)Y + \left(\frac{1}{2}\ln A - 1\right)Y^2 + \left(\frac{1}{2} - \frac{1}{6}\ln A\right)Y^3$
INCOMPLETE
$F(X) \simeq \ln A + (1 - \ln A)(X - \ln A) + \frac{1}{2}(-2 + \ln A)(X - \ln A)^2 + \frac{1}{6}(3 - \ln A)(X - \ln A)^3 = \left(1 + \ln A - \frac{1}{2}\ln^3 A + \frac{1}{2}\ln^2 A\right)X - \left(1 + \ln A - \frac{1}{2}\left(\ln^2 A\right)\right)X^2 + \left(\frac{1}{2} - \frac{1}{6}\ln A\right)X^3 + \frac{1}{6}\ln^4 A$

7.15

7.16

7.17

7.18

7.19

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7.21

7.22

7.23

7.24

7.25

7.26

7.27

7.28

8

8.1

$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} \frac{dq_i}{dt} \right) = \sum_i \left(-\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} \right) = 0$

8.2

$H = \frac{p^2}{2mL^2} + mgL\,(1 - \cos x)$. In this case we have $q \equiv x$. Eqn 8.5 gives : $\frac{dp}{dt} = -\frac{\partial H}{\partial \theta} = -mgL \sin x = mL^2 \dfrac{d^2 x}{dt^2} \Rightarrow \dfrac{dv}{dt} = -\omega^2 \sin x$ with $v = \dfrac{dx}{dt}$ and $\omega = \sqrt{\frac{g}{L}}$.

8.3

The Jacobian Matrix is given by: $J(x, v) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{pmatrix}$ which, evaluated at $x = \pi$, $v = 0$ gives $J(\pi, 0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix}$. The characteristic polynomial is $\lambda^2 - \omega^2 = 0$ which gives $\lambda = \pm\omega \implies$ at $x = \pi$, $v = 0$ we have a saddle point.

8.4

Start from $E - mgL = \frac{mL^2}{2} \left(\frac{dx}{dt}\right)^2 - mgL \cos x$. Let x_0 be the maximum value of x, which satisfies $\cos x_0 = -\frac{E - mgL}{mgL}$. Hence $\frac{L}{2g} \left(\frac{dx}{dt}\right)^2 = \cos x - \cos x_0$. Solving for t : $dt = \sqrt{\frac{L}{2g}} \frac{dx}{\sqrt{\cos x - \cos x_0}}$. Integrate from 0 to x_0 to get 1/4 of the period T which can be written as $T = 4\sqrt{\frac{L}{2g}} \int_0^{x_0} \frac{dx}{\sqrt{\cos x - \cos x_0}} = 2\sqrt{\frac{L}{g}} \int_0^{x_0} \frac{dx}{\sqrt{\sin^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x_0}}$. Make the substitution $\sin y = \sin^2 \frac{1}{2}x / \sin^2 \frac{1}{2}x_0$ to convert it to $T = 4\sqrt{\frac{L}{g}} K(\sin \frac{1}{2}x_0)$ where $K(s) = \int_0^{\pi/2} \frac{ds}{\sqrt{1 - k^2 \sin^2 s}}$ is the complete elliptic integral of the first kind. Since $\sin \frac{1}{2}x_0 = \sqrt{\frac{1 - \cos x_0}{2}} = \sqrt{\frac{E}{2mgL}}$, use these formulas to get $T_{E/mgL}$:

$T_{0.1} = 6.364\sqrt{\frac{L}{g}}$, $T_{1.0} = 7.416\sqrt{\frac{L}{g}}$, $T_{2.0} = \infty$.

8.5

In this case we have $q_1 \equiv x$, $q_2 \equiv y$, $p_1 \equiv v$, $p_2 \equiv w$. Using eqns 8.4 and 8.5 we get : $\frac{dp_1}{dt} = \frac{dv}{dt} = -\frac{\partial H}{\partial q_1} = -\frac{\partial H}{\partial x} = -\omega^2 \sin x + Ay$, $\frac{dp_2}{dt} = \frac{dw}{dt} = -\frac{\partial H}{\partial q_2} = -\frac{\partial H}{\partial y} = -\Omega^2 y + Ax$, $\frac{dq_1}{dt} = \frac{dx}{dt} = \frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial v} = v$ and $\frac{dq_2}{dt} = \frac{dy}{dt} = \frac{\partial H}{\partial p_2} = \frac{\partial H}{\partial w} = w$.

8.6

We have $q_1 \equiv x$, $q_2 \equiv y$, $p_1 \equiv v$, $p_2 \equiv w$. Use eqns 8.4 and 8.5 to get : $\frac{dq_1}{dt} = \frac{dx}{dt} = \frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial v} = v$, $\frac{dq_2}{dt} = \frac{dy}{dt} = \frac{\partial H}{\partial p_2} = \frac{\partial H}{\partial w} = w$, $\frac{dp_1}{dt} = \frac{dv}{dt} = -\frac{\partial H}{\partial q_1} = -\frac{\partial H}{\partial x} = -2xy - x$ and $\frac{dp_2}{dt} = \frac{dw}{dt} = -\frac{\partial H}{\partial q_2} = -\frac{\partial H}{\partial y} = -y - x^2 + y^2$.

8.7

LOOK UP FORMULA IN FORD'S REFERENCE

8.8

First note that $x = \frac{\chi + \psi}{2}$, $y = \frac{\chi - \psi}{2}$. Eqn 8.18 may be written as $2\frac{dv}{dt} = 2\frac{d^2x}{dt^2} = \frac{d^2\chi}{dt^2} + \frac{d^2\psi}{dt^2} = -\chi - \psi - \chi^2 + \psi^2$ and eqn 8.19 (with y^2 replaced by $-y^2$) as $2\frac{dw}{dt} = 2\frac{d^2y}{dt^2} = \frac{d^2\chi}{dt^2} - \frac{d^2\psi}{dt^2} = -\chi + \psi - \chi^2 - \psi^2$. Solve for $\frac{d^2\chi}{dt^2}$ and $\frac{d^2\psi}{dt^2}$ to obtain eqns 8.21 and 8.22.
SOLUTION??

8.9

Use eqn 5.24 with $f = y$, $g = -x + yz$ and $h = 1 - y^2$, from which it follows that $\frac{1}{V}\frac{dV}{dt} = \lim_{T\to\infty} \frac{1}{T}\int_0^T z dt = \langle z \rangle$

8.10

If we make the change of variables $t \to -t$, $z \to -z$ and $y \to -y$ we still get the same equations. Therefore the system is time reversible.

8.11

Since performing the change of variables $t \to -t$, $z \to -z$ and $x \to -x$ we still get the same equations, the system is time reversible. Also , using eqn 5.24 together with $f = -y$, $g = x + z$ and $h = xz + 3y^2$ we get for the time-reversed system : $\frac{1}{V}\frac{dV}{dt} = \lim_{T\to\infty} \frac{1}{T}\int_T^0 x dt = -\langle x \rangle$.

8.12

Use $\dfrac{dx}{dt} = y$, $\dfrac{dy}{dt} = -x{+}yz$ and $\dfrac{dz}{dt} = 1{-}y^2$ .Since $\dfrac{dy}{dt} = \dfrac{d^2x}{dt^2} = -x{+}\dfrac{dx}{dt}z \Rightarrow$ $z = \left(\dfrac{d^2x}{dt^2} + x\right)/\dfrac{dx}{dt}$. Taking the derivative of z with respect to t we get $\dfrac{dz}{dt} = \left(\dfrac{d^3x}{dt^3} + \dfrac{dx}{dt}\right)/\dfrac{dx}{dt} - \dfrac{d^2x}{dt^2}\left(\dfrac{d^2x}{dt^2} + x\right)/\left(\dfrac{dx}{dt}\right)^2 = 1 - \left(\dfrac{dx}{dt}\right)^2$. Simplifying the last result yields eqn 8.31.

8.13

The differential equation for the simple pendulum can be written as $\dfrac{dx}{dt} = y = f(x,y)$ and $\dfrac{dy}{dt} = k\sin x = g(x,y)$. For the case of the Leapfrog method see solutions to problems 8.20 and 8.21. For the Runge-Kutta method use $k_{1xn} = Y_n$, $k_{1yn} = k\sin X_n$, $k_{2xn} = Y_n + \frac{1}{2}k\sin X_n$ and $k_{2yn} = k\sin(X_n + \frac{1}{2}Y_n)$. Then $X_{n+1} = X_n + Y_n + \frac{1}{2}k\sin X_n$ and $Y_{n+1} = Y_n + k\sin(X_n + \frac{1}{2}Y_n)$. The Jacobian of this map has determinant $\det J = \left(1 + \frac{1}{2}k\cos X_n\right)\left(1 + \frac{1}{2}k\cos(X_n + \frac{1}{2}Y_n)\right) - k\sin(X_n + \frac{1}{2}Y_n) \neq 1$. Hence the map in this case is not simplectic (area-preserving).

8.14

Use the Jacobian of the map is given at problem 5.12. whose determinant is $\det J = \cos^2\alpha + \sin^2\alpha = 1$.

8.15

The Hénon area-preserving quadratic map is not invertible.

8.16

The Jacobian is given by $J(X,Y) = \begin{pmatrix} a_2 + 2a_4X + a_5Y & a_3 + a_5X + 2a_6Y \\ 1 & 0 \end{pmatrix}$
The map is area preserving if $|\det J| = |-a_3 - a_5X - 2a_6Y| = 1$ for all X and Y. This is the case only if $|a_3| = 1$ and $a_5 = a_6 = 0$.

8.17

The Jacobian is $J(X,Y) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ whose eigenvalues are $\lambda_\pm = \frac{3\pm\sqrt{5}}{2}$ with eigenvectors $v_\pm = \begin{pmatrix} 1 \\ \lambda_\pm - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1\pm\sqrt{5}}{2} \end{pmatrix}$ Since $v_+ \cdot v_- = 0$ we may conclude that the two manifolds are perpendicular to each other. The unstable manifold corresponds to λ_+. The tangent of the angle of v_+ with the x-axis is given by the ratio of the y over the x component of the vector which is obviously equal to the golden mean.

8.18

As indicated in the text on page 169 if we start from $(-1,3) \to (-1,-1) \to (3,-1) \to (5,3) \to (3,5) \to (-1,3)$ which is a period 5 cycle.

8.19

Use the Jacobian of the map is given at problem 5.12. whose determinant is $\det J = \cos^2\alpha + \sin^2\alpha = 1$.

8.20

he differential equation for the simple pendulum $\frac{d^2x}{dt^2} + k\sin x = 0$ can be transformed to a system of differential equations :$\frac{dx}{dt} = y = f(x,y)$ and $\frac{dy}{dt} = k\sin x = g(x,y)$. The Leapfrog method in the y-variable can be written as $Y_{n+1} = Y_n + hg(X_n, Y_n)$ and $X_{n+1} = X_n + hf(X_n, Y_{n+1})$. Since we are dealing with angles we may take the equations modulo 2π and if we use $h = 1$ we finally obtain eqns 8.40 and 8.41.

8.21

Use the Jacobian from the solution to problem 5.12 and calculate its determinant which equals $\det J = 1 + k\cos X - k\cos X = 1$. Therefore the map is area-preserving.

8.22

The fixed points satisfy $Y_n(\mathsf{mod}2\pi) = 0$ and $k\sin X_n(\mathsf{mod}2\pi) = 0$. The fixed points are $Y^* = 0$ and $X^* = 0$ or π. The Jacobian of the map is given in the solution of problem 5.12 with characteristic polynomial $\lambda^2 - \lambda(2 + k\cos X) + 1 = 0$. When $X^* = 0$ we get $\lambda^2 - (2+k)\lambda + 1 = 0$ whose solution is $\lambda = 1 + \frac{1}{2}k \pm \frac{1}{2}k\sqrt{\left(\frac{4}{k}+1\right)}$. So for all k we have a saddle point.When $X^* = \pi$ we have the equation $\lambda^2 - (2-k)\lambda + 1 = 0$ whose solution is $\lambda = 1 - \frac{1}{2}k + \frac{1}{2}k\sqrt{\left(1 - \frac{4}{k}\right)}$. In this case we have an unstable spiral point for $0 < k < 2$, a stable spiral for $2 < k < 4$ and a stable fixed point for $k > 4$.

8.23

Start with $X_0 = 0$ and $Y_0 = \pi$. The first iteration will give $Y_1 = \pi(\mathsf{mod}2\pi) = \pi$ and $X_1 = \pi(\mathsf{mod}2\pi) = \pi$. The second iteration will give $Y_2 = \pi(\mathsf{mod}2\pi) = \pi$ and $X_2 = 2\pi(\mathsf{mod}2\pi) = 0$ which correspond to the initial point (X_0, Y_0) .

8.24

One solution is $(0, \frac{2\pi}{3}) \to (\frac{2\pi}{3}, \frac{2\pi}{3}) \to (\frac{4\pi}{3}, \frac{2\pi}{3}) \to (0, \frac{2\pi}{3})$

8.25

The sum of the eigenvalues equals to the logarithm of the determinant which in this case equals to $\log|\det J| = \log|b(1+k\cos X) - bk\cos X| = \log|b|$.

8.26

In order to show that Q is conserved we have to show that $A = (X_n^2 + X_{n-1}^2 + X_{n-2}^2 - X_n X_{n-1} X_{n-2}) - (X_{n-1}^2 + X_{n-2}^2 + X_{n-3}^2 - X_{n-1} X_{n-2} X_{n-3}) = 0$. A simplifies to $A = X_n^2 - X_{n-3}^2 - X_{n-1} X_{n-2}(X_n - X_{n-3}) = X_n^2 - (X_{n-1} X_{n-2} - X_n)^2 - X_{n-1} X_{n-2}(2X_n - X_{n-1} X_{n-2}) = 0$. Hence Q is conserved.

9

10

10.1

?

10.2

Assume the polynomial to be of the form $y = az^2 + bz + c$ and passing through the points $(-1, X_{n-2})$, $(0, X_{n-1})$ and $(1, X_n)$. Solving the resulting system for a, b and c we get $a = \frac{1}{2}X_n + \frac{1}{2}X_{n-2} - X_{n-1}$, $b = \frac{1}{2}X_n - \frac{1}{2}X_{n-2}$ and $c = X_{n-1}$. Using these coefficients set z equal to 2 to find X_{n+1}.

10.3

Assume the polynomial to be of the form $y = az^3 + bz^2 + cz + d$ and passing trough the points $(-2, X_{n-3})$, $(-1, X_{n-2})$, $(0, X_{n-1})$ and $(1, X_n)$.Solving the resulting system for a, b, c and d we get $a = -\frac{1}{2}X_{n-1} + \frac{1}{6}X_n + \frac{1}{2}X_{n-2} - \frac{1}{6}X_{n-3}$, $b = -X_{n-1} + \frac{1}{2}X_n + \frac{1}{2}X_{n-2}$, $c = \frac{1}{2}X_{n-1} + \frac{1}{3}X_n - X_{n-2} + \frac{1}{6}X_{n-3}$ and $d = X_{n-1}$. Using these coefficients set z equal to 2 to find X_{n+1}.

10.4

Assume the polynomial to be of the form $y = az^4 + bz^3 + cz^2 + dz + e$ and passing through the points $(-2, X_{n-4})$, $(-1, X_{n-3})$, $(0, X_{n-2})$, $(1, X_{n-1})$ and $(2, X_n)$. Solving the resulting system for a, b, c, d and e we get $a = \frac{1}{4}X_{n-2} + \frac{1}{24}X_n + \frac{1}{24}X_{n-4} - \frac{1}{6}X_{n-1} - \frac{1}{6}X_{n-3}$, $b = \frac{1}{12}X_n - \frac{1}{12}X_{n-4} - \frac{1}{6}X_{n-1} + \frac{1}{6}X_{n-3}$, $c = -\frac{5}{4}X_{n-2} - \frac{1}{24}X_n - \frac{1}{24}X_{n-4} + \frac{2}{3}X_{n-1} + \frac{2}{3}X_{n-3}$, $d = -\frac{1}{12}X_n + \frac{1}{12}X_{n-4} + \frac{2}{3}X_{n-1} - \frac{2}{3}X_{n-3}$ and $e = X_{n-2}$. Using these coefficients set z equal to 3 to find X_{n+1}.

10.5

Assume the polynomial to be of the form $y = az^5 + bz^4 + cz^3 + dz^2 + ez + f$ and passing through the points $(-3, X_{n-5})$, $(-2, X_{n-4})$, $(-1, X_{n-3})$, $(0, X_{n-2})$, $(1, X_{n-1})$ and $(2, X_n)$. Solving the resulting system for a, b, c, d, e and f we get: $a = \frac{1}{12}X_{n-2} - \frac{1}{12}X_{n-3} - \frac{1}{24}X_{n-1} + \frac{1}{120}X_n - \frac{1}{120}X_{n-5} + \frac{1}{24}X_{n-4}$,
$b = \frac{1}{4}X_{n-2} + \frac{1}{24}X_n + \frac{1}{24}X_{n-4} - \frac{1}{6}X_{n-1} - \frac{1}{6}X_{n-3}$,
$c = -\frac{5}{12}X_{n-2} + \frac{7}{12}X_{n-3} + \frac{1}{24}X_{n-1} + \frac{1}{24}X_n + \frac{1}{24}X_{n-5} - \frac{7}{24}X_{n-4}$,
$d = -\frac{5}{4}X_{n-2} + \frac{2}{3}X_{n-3} + \frac{2}{3}X_{n-1} - \frac{1}{24}X_n - \frac{1}{24}X_{n-4}$,
$e = -\frac{1}{20}X_n - \frac{1}{30}X_{n-5} + \frac{1}{4}X_{n-4} - X_{n-3} + \frac{1}{3}X_{n-2} + \frac{1}{2}X_{n-1}$ and $f = X_{n-2}$. Using these coefficients set z equal to 3 to find X_{n+1}.

10.6

The values to be predicted are:
$X_{11} = \sin 11 = -0.99999$, $X_{12} = \sin 12 = -0.53657$ and $X_{13} = \sin 13 = 0.$
42017 Using eqn 10.1 we obtain:
$X_{11} = -1.8791$, $X_{12} = -3.5931$ and $X_{13} = -5.686$
Using eqn 10.2:
$X_{11} = -1.5002$, $X_{12} = -2.4564$ and $X_{13} = -3.4126$
Using eqn 10.3:
$X_{11} = -1.3483$, $X_{12} = -1.47$ and $X_{13} = -0.3784$
Using eqn 10.4:
$X_{11} = -0.51206$, $X_{12} = 2.7113$ and $X_{13} = 12.166$
Using eqn 10.5:
$X_{11} = -0.23111$, $X_{12} = 4.397$ and $X_{13} = 18.035$

10.7

Using the original map, the first 6 points are: $X_1 = 0.1$, $X_2 = 0.36$, $X_3 = 0$ .9216, $X_4 = 0.28901$, $X_5 = 0.82193$ and $X_6 = 0.58544$. Knowing the first 3 points we may use eqn 10.1 to get $X_4 = 1.4832$, $X_5 = 2.0448$ and $X_6 = 2.6064$. Applying eqn 10.2 to the same set of 3 points we get $X_4 = 1.7848$, $X_5 = 2.9496$ and $X_6 = 4.416$.

10.8

Set $j = m = 1$ in eqns 10.7 and 10.8 and the result follows immediately.

10.9

$a_1 = \dfrac{X_4X_3 + X_3X_2 + X_2X_1}{X_3^2 + X_2^2 + X_1^2} = \dfrac{32+8+2}{16+4+1} = 2$. Hence $X_{n+1} = 2X_n$.

10.10

$a_1 = \dfrac{X_2X_1}{X_1^2} = 5$. Hence $X_{n+1} = 5X_n$ so $X_3 = 25$, $X_4 = 125$ and $X_5 = 625$

10.11

11

12

12.1

$D = \log 3 / \log 5 = 0.6826$

12.2

$D = \log 3 / \log 7 = 0.6826$

12.3

$D = \log 4 / \log 3 = 1.262$

12.4

Answers given in section 11.4

12.5

Answer given in section 11.5

12.6

At the n-th stage of the construction we need a minimum of 2^m boxes with $L = (1/3)^m$, so $D_0 = -\frac{-\log 2^m}{\log(1/3)^m} = \frac{\log 2}{\log 3} = 0.6309$

12.7

12.8

12.9

By the definition of $C(r)$ we have $C(0) = 0$. As $r \to \infty$ all Heaviside functions evaluate to 1 since a circle with infinite radius contains all X_n's. In that case the double sum evaluates to $\sum_{j=1}^{N} (N-j) = 1 + 2 + ... + N - 1 = \frac{N(N-1)}{2}$. This gives $C(\infty) = 1$.

12.10

$C(r) = \int_0^r f(r)dr$, where $f(r)$ is the distribution of points on the real line.

12.11

Referring to problem 12.10,let $f(r) = 1$. In this case the integral is performed on a surface so $C(r) = \int_0^{2\pi}\int_0^{r} r dr d\theta = \pi r^2$. Hence $D_2 = \frac{d\log(\pi r^2)}{d\log r} = \frac{d[2\log r + \log \pi]}{d\log r} = 2$

12.12

12.13

12.14

12.15

The answers for $D = 1, 2, ..., 10$ are $252, 631, 1585, 3982, 10000, 25119, 63096, 1.58\times 10^5, 3.98\times 10^5,\ 10^6$ respectively.

12.16

12.17

12.18

$D = \frac{\log\frac{494}{298}}{\log\frac{5}{3}} = 0.989$

12.19

$D = \frac{\log\frac{244036}{88804}}{\log\frac{5}{3}} = 1.979$

12.20

$D = \frac{\log\frac{5(a+b)-2}{3(a+b)-2}}{\log\frac{5}{3}}$